命題 1.1 補足

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$，$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$，$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ として，実際に計算してみよう．それぞれ実数のどのような性質が用いられているかを観察されたい．

(a)
$$\begin{aligned}
(A+B)+C &= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \left\{ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right\} \\
&= A+(B+C)
\end{aligned}$$

実数の和に関する結合法則を用いている．

(b)
$$\begin{aligned}
A+B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \\
&= \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{pmatrix} \\
&= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= B+A
\end{aligned}$$

実数の和に関する交換法則を用いている．

(c)
$$\begin{aligned}
(AB)C &= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}b_{11}+a_{12}b_{21})c_{11} + (a_{11}b_{12}+a_{12}b_{22})c_{21} & (a_{11}b_{11}+a_{12}b_{21})c_{12} + (a_{11}b_{12}+a_{12}b_{22})c_{22} \\ (a_{21}b_{11}+a_{22}b_{21})c_{11} + (a_{21}b_{12}+a_{22}b_{22})c_{21} & (a_{21}b_{11}+a_{22}b_{21})c_{12} + (a_{21}b_{12}+a_{22}b_{22})c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21}) + a_{12}(b_{21}c_{11}+b_{12}c_{21}) & a_{11}(b_{11}c_{12}+b_{12}c_{22}) + a_{12}(b_{21}c_{12}+b_{22}c_{22}) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21}) + a_{22}(b_{21}c_{11}+b_{22}c_{21}) & a_{21}(b_{11}c_{12}+b_{12}c_{22}) + a_{22}(b_{21}c_{12}+b_{22}c_{22}) \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{12}c_{21} & b_{21}c_{12}+b_{22}c_{22} \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left\{ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right\}$$
$$= A(BC)$$

実数の分配法則，和に関する結合法則・交換法則，積に関する結合法則を用いている．3 行目から 5 行目にかけてをシグマ記号での表記

$$\sum_{j=1}^{2} \left(\sum_{k=1}^{2} a_{ik} b_{kj} \right) c_{jl} = \sum_{j=1}^{2} \sum_{k=1}^{2} a_{ik} b_{kj} c_{jl} = \sum_{k=1}^{2} a_{ik} \left(\sum_{j=1}^{2} b_{kj} c_{jl} \right) \qquad (i, l = 1, 2)$$

と比較しておこう．

(d) $A(B+C) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left\{ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right\}$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22}) \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} + a_{11}c_{11} + a_{12}b_{21} + a_{12}c_{21} & a_{11}b_{12} + a_{11}c_{12} + a_{12}b_{22} + a_{12}c_{22} \\ a_{21}b_{11} + a_{21}c_{11} + a_{22}b_{21} + a_{22}c_{21} & a_{21}b_{12} + a_{21}c_{12} + a_{22}b_{22} + a_{22}c_{22} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} + \begin{pmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$
$$= AB + AC$$

実数の分配法則，和に関する結合法則・交換法則を用いている．

(e) (d) と同様なので省略．

(f) $(st)A = \begin{pmatrix} (st)a_{11} & (st)a_{12} \\ (st)a_{21} & (st)a_{22} \end{pmatrix}$

$$= \begin{pmatrix} s(ta_{11}) & s(ta_{12}) \\ s(ta_{21}) & s(ta_{22}) \end{pmatrix}$$
$$= s \begin{pmatrix} ta_{11} & ta_{12} \\ ta_{21} & ta_{22} \end{pmatrix}$$
$$= s(tA)$$

実数の積に関する結合法則を用いている．

(g) $(s+t)A = \begin{pmatrix} (s+t)a_{11} & (s+t)a_{12} \\ (s+t)a_{21} & (s+t)a_{22} \end{pmatrix}$

$$= \begin{pmatrix} sa_{11} + ta_{11} & sa_{12} + ta_{12} \\ sa_{21} + ta_{21} & sa_{22} + ta_{22} \end{pmatrix}$$
$$= \begin{pmatrix} sa_{11} & sa_{12} \\ sa_{21} & sa_{22} \end{pmatrix} + \begin{pmatrix} ta_{11} & ta_{12} \\ ta_{21} & ta_{22} \end{pmatrix}$$
$$= sA + tA$$

実数の分配法則を用いている．

(h) $t(A+B) = t \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

$$= \begin{pmatrix} t(a_{11}+b_{11}) & t(a_{12}+b_{12}) \\ t(a_{21}+b_{21}) & t(a_{22}+b_{22}) \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} ta_{11}+tb_{11} & ta_{12}+tb_{12} \\ ta_{21}+tb_{21} & ta_{22}+tb_{22} \end{pmatrix} \\
&= \begin{pmatrix} ta_{11} & ta_{12} \\ ta_{21} & ta_{22} \end{pmatrix} + \begin{pmatrix} tb_{11} & tb_{12} \\ tb_{21} & tb_{22} \end{pmatrix} \\
&= tA + tB
\end{aligned}
$$

実数の分配法則を用いている．

(i)
$$
\begin{aligned}
t(AB) &= t\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \\
&= \begin{pmatrix} t(a_{11}b_{11}+a_{12}b_{21}) & t(a_{11}b_{12}+a_{12}b_{22}) \\ t(a_{21}b_{11}+a_{22}b_{21}) & t(a_{21}b_{12}+a_{22}b_{22}) \end{pmatrix} \\
&= \begin{pmatrix} ta_{11}b_{11}+ta_{12}b_{21} & ta_{11}b_{12}+ta_{12}b_{22} \\ ta_{21}b_{11}+ta_{22}b_{21} & ta_{21}b_{12}+ta_{22}b_{22} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(tA)B &= \begin{pmatrix} ta_{11} & ta_{12} \\ ta_{21} & ta_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} ta_{11}b_{11}+ta_{12}b_{21} & ta_{11}b_{12}+ta_{12}b_{22} \\ ta_{21}b_{11}+ta_{22}b_{21} & ta_{21}b_{12}+ta_{22}b_{22} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A(tB) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} tb_{11} & tb_{12} \\ tb_{21} & tb_{22} \end{pmatrix} \\
&= \begin{pmatrix} ta_{11}b_{11}+ta_{12}b_{21} & ta_{11}b_{12}+ta_{12}b_{22} \\ ta_{21}b_{11}+ta_{22}b_{21} & ta_{21}b_{12}+ta_{22}b_{22} \end{pmatrix}
\end{aligned}
$$

よって $t(AB) = (tA)B = A(tB)$．実数の分配法則，積に関する結合法則・交換法則を用いている．